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# String center of mass operator and its effect on BRST cohomology

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**Abstract**

We consider the theory of bosonic closed strings on the flat background $\mathbb{R}^{25,1}$. We show how the BRST complex can be extended to a complex where the string center of mass operator, $x_0^\mu$, is well defined. We investigate the cohomology of the extended complex. We demonstrate that this cohomology has a number of interesting features. Unlike in the standard BRST cohomology, there is no doubling of physical states in the extended complex. The cohomology of the extended complex is more physical in a number of of aspects related to the zero-momentum states. In particular, we show that the ghost number one zero-momentum cohomology states are in one to one correspondence with the generators of the global symmetries of the background *i.e.*, the Poincaré algebra.

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# 1 Introduction and summary

As it was argued in ref. [1], the gauge parameters that include the zero mode of the $X^\mu$ operator have to be considered in order to prove the complete dilaton theorems. If we allow $X^\mu$ to appear in gauge parameters, it is natural to allow it to appear in the physical states as well. This calls for an extended version of the BRST complex where the zero mode of $X^\mu$ or, in other words, the string center of mass operator $x_0^\mu$ is well defined.

In this paper we will consider the closed bosonic string with a flat non-compact target space $\mathbb{R}^{25,1}$. We will define the extended complex simply as a tensor product of the BRST complex with the space of polynomials of $D = 26$ variables. Our main objective is to calculate the semi-relative cohomology of this complex.

When we add new vectors to a complex, two phenomena may occur in cohomology. First, some vectors that used to represent nontrivial cohomology classes may become trivial, and second, some new cohomology states may appear. Our original motivation [1] to use the extended complex was that the graviton trace $\mathcal{G}$, BRST-physical state, was trivial in the extended complex. As we will see, the extended complex provides many more examples of this kind. We will show that only one out of $D^2 + 1$ ghost number two zero-momentum BRST-physical states remains non-trivial in the extended complex. In the BRST complex, the spectrum of non-zero momentum physical states is doubled due to the presence of the ghost zero modes. We will see that there is no such doubling in the extended complex: only ghost number two states survive and all the ghost number three states become trivial[1]. Returning to the second phenomenon, the appearance of new physical states, we will be able to show that all such states can be obtained from the old ones by differentiation with respect to the continuous momentum parameter along the corresponding mass shells. In this sense no new physical state will appear.

Zero momentum states will require special attention and the calculation of the cohomology of the zero momentum extended complex is technically the most difficult part of this work. We will find that the ghost number one discrete states at zero momentum correctly describe the global symmetries

[1]The observation that ghost number three states are physical only in the case of finite space time volume was made in the book by Green, Schwarz, and Witten [2]. In the finite space time volume the momentum is discrete and $x_0^\mu = -i\partial/\partial p_\mu$ cannot be defined.

(Poincaré group) of the background.

The analysis presented in this paper shows that the semi-relative cohomology of the extended complex correctly describes the physics of the closed bosonic string around the flat $D = 26$ background. The arguments that lead to this conclusion are the following:

1. Ghost number two non-discrete physical states are the same as in the BRST complex up to infinitesimal Lorentz transformations. In this sense the extended complex is as good as BRST (section 5.1).

2. There is no doubling of the physical states,—ghost number three BRST-physical states are trivial in the extended complex (section 4.2).

3. There is only one zero-momentum physical state at ghost number two which can be represented by the ghost dilaton (section 4.3 and ref. [1]).

4. Ghost number one discrete states are in one to one correspondence with the generators of the Poincaré group (section 4.3).

The paper is organized as follows. In section 2 we start by describing the extended complex and the nilpotent operator $\widehat{Q}$. We define a cohomology problem for the extended complex and explain what we are going to learn about its structure. In section 3 we formulate a simplified version of the problem in which we replace the closed string BRST complex by its chiral part. Section 3 is devoted to a detailed analysis of the cohomology of this complex. In section 4 we investigate the (semi-relative) cohomology of the full extended complex using the same methods as in section 3. At the end of section 3 and section 4 we will formulate two theorems which summarize our results on the structure of the cohomology of the chiral and the full extended complexes. We present a detailed analysis of the Lorentz group action on the cohomology of the extended complex in section 5. For the sake of completeness we add two appendices: Appendix A, where we review some basic algebraic facts which we use to calculate the cohomology and Appendix B, where we prove the results which are necessary to calculate the cohomology at zero momentum.

# 2 Extended complex

When we describe a string propagating in flat uncompactified background, it seems natural to let the string center of mass operator $x_0$ act on the state space of the theory. This operator must satisfy the Heisenberg commutation relation

$$[\alpha^\mu_n, x^\nu_0] = -i\,\delta_{n,0}\,\eta^{\mu\nu}, \tag{2.1}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. In order to incorporate an operator with such properties we define an extended Fock space as a tensor product of an ordinary Fock space with the space of polynomials of $D$ variables:

$$\widehat{\mathcal{F}}_p(\alpha,\overline{\alpha}) = \mathbb{C}[x^0,\ldots,x^{D-1}] \otimes \mathcal{F}_p(\alpha,\overline{\alpha}). \tag{2.2}$$

All operators except $\alpha^\mu_0$ act only on the second factor which is an ordinary Fock space, $x^\mu_0$ operators act by multiplying the polynomials by the corresponding $x^\mu$ and, finally, the action of $\alpha_0$ is defined by

$$\alpha^\mu_0 = 1 \otimes p^\mu - i\eta^{\mu\nu}\frac{\partial}{\partial x^\nu} \otimes 1. \tag{2.3}$$

Note that in the extended Fock module the action of $\alpha^\mu_0$ does not reduce to the multiplication by $p^\mu$.

The extended Fock module is also a module over Virasoro algebra or, strictly speaking, over a tensor product of two Virasoro algebras corresponding to the left and right moving modes. The generators are given by the usual formulae

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m \eta_{\mu\nu}\ :\alpha^\mu_m\,\alpha^\nu_{n-m}:, \\ \overline{L}_n &= \tfrac{1}{2}\textstyle\sum_m \eta_{\mu\nu}\ :\overline{\alpha}^\mu_m\,\overline{\alpha}^\nu_{n-m}: . \end{aligned} \tag{2.4}$$

The central charge of the extended Fock module is 26, the same as that of $\mathcal{F}_p(\alpha,\overline{\alpha})$, and we can use it to construct a complex with a nilpotent operator $\widehat{Q}$ (see refs. [2, 3]). Following the standard procedure we define the extended complex as a tensor product of the extended Fock module with the ghost module $\mathcal{F}(b,c,\overline{b},\overline{c})$

$$\widehat{V}_p = \mathcal{F}(b,c,\overline{b},\overline{c}) \otimes \widehat{\mathcal{F}}_p(\alpha,\overline{\alpha}) \tag{2.5}$$

and introduce the nilpotent operator $\widehat{Q}$ as

$$\widehat{Q} = \sum_n c_n \widehat{L}_{-n} - \frac{1}{2}\sum_{m,n}(m-n)\ :c_{-m}c_{-n}b_{m+n}: + \text{a.h.}, \tag{2.6}$$

where we put a hat over Virasoro generators in order to emphasize that they are acting on the extended Fock space.

We can alternatively describe the extended complex as a tensor product of the BRST complex $V_p$ with the space of polynomials $\mathbb{C}[x^0 \cdots x^{D-1}]$:

$$\widehat{V}_p = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes V_p, \tag{2.7}$$

and express the nilpotent operator $\widehat{Q}$ in terms of the BRST operator $Q$ as follows

$$\widehat{Q} = 1 \otimes Q - i\frac{\partial}{\partial x^\mu} \otimes \sum_n (c_n \alpha^\mu_{-n} + \overline{c}_n \overline{\alpha}^\mu_{-n}) - \Box \otimes c_0^+, \tag{2.8}$$

where $c_0^+ = (c_0 + \overline{c}_0)/2$, and $\Box = \eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\mu}$.

So far we have constructed an extended complex $\widehat{V}_p = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes V_p$ equipped with a nilpotent operator $\widehat{Q}$ given by Eq. (2.8). One can easily check that Eq. (2.8) does define a nilpotent operator using $Q^2 = 0$ and commutation relations between $\alpha^\mu_n$ operators.

## 2.1 Cohomology of the extended complex

Our major goal is to calculate the cohomology of the operator $\widehat{Q}$ acting on the extended complex $\widehat{V}_p$ for different values of the momentum $p$. More precisely, we will be looking for the so-called semi-relative cohomology, which is the space of vectors annihilated by $\widehat{Q}$ and $b_0^- = b_0 - \overline{b}_0$ modulo the image of $\widehat{Q}$ acting on the vectors annihilated by $b_0^-$ (see refs. [4, 5, 6, 7]).

In mathematical literature it is called the semi-infinite cohomology of the $Vir \times Vir$ algebra relative to the sub-algebra $\mathcal{L}_0^-$ generated by the central charge and $L_0^- = L_0 - \overline{L}_0$ with the values in the extended Fock module $\widehat{\mathcal{F}}_p$, and is denoted by $\mathcal{H}(Vir \times Vir, \mathcal{L}_0^-, \widehat{\mathcal{F}}_p)$ (see refs. [8, 9]). We denote this cohomology by $H_S(\widehat{Q}, \widehat{V}_p)$.

Before we start the calculations, let us describe what kind of information about the cohomology we want to obtain. Ordinarily, we are looking for the dimensions of the cohomology spaces at each ghost number. In the presence of the $x_0$ operator these spaces are likely to be infinite dimensional (since multiplication by $x$ does not change the ghost number). In order to extract reasonable information about the cohomology we have to use an additional grading by the degree of the polynomials in $x$. The operator $\widehat{Q}$ mixes vectors of different degrees and we can not a priori expect the cohomology states to

be represented by homogeneous polynomials.[2] Instead of the grading on $\widehat{V}$ we have to use a decreasing filtration

$$\cdots \supset F_{-k}\widehat{V} \supset F_{-k+1}\widehat{V} \supset \cdots \supset F_0\widehat{V} = V, \tag{2.1.1}$$

where $F_{-k}\widehat{V}$ is the subspace of $\widehat{V}$ consisting of the vectors with $x$ degree less or equal to $k$. The operator $\widehat{Q}$ respects this filtration in a sense that it maps each subspace $F_r\widehat{V}$ to itself:

$$\widehat{Q}: \ F_r\widehat{V} \to F_r\widehat{V}. \tag{2.1.2}$$

This allows us to define a filtration on the cohomology of $\widehat{Q}$. By definition $F_{-k}H(\widehat{Q},\widehat{V})$ consists of the cohomology classes which can be represented by vectors with the $x$-degree less or equal to $k$. Although as we mentioned above there is no $x$-grading on the cohomology space, we can define a graded space which is closely related to it. We define

$$\mathrm{Gr}_r H(\widehat{Q},\widehat{V}) = F_r H(\widehat{Q},\widehat{V})/F_{r+1}H(\widehat{Q},\widehat{V}). \tag{2.1.3}$$

By definition $\mathrm{Gr}_r H(\widehat{Q},\widehat{V})$ (for $r \leq 0$) consists of the cohomology classes which can be represented by a vector with $x$-degree $-r$ but not $-r+1$. These spaces carry a lot of information about the cohomology structure. For example if we know the dimensions of $\mathrm{Gr}_r$ for $r = 0, -1, \ldots, -k$ we can find the dimension of $F_{-k}$ as their sum. On the other hand the knowledge of representatives of $\mathrm{Gr}_r$ states is not enough to find the representatives of cohomology classes. This is so because $\mathrm{Gr}_r$ spaces contain information only about the leading in $x$ terms of the cocycles of $\widehat{Q}$.

To find the graded space $\mathrm{Gr}H = \bigoplus \mathrm{Gr}_r H$ we will use the machinery known as the method of spectral sequences. The idea is to build a sequence of complexes $E_n$ with the differentials $d_n$ such that $E_{n+1} = H(d_n, E_n)$ which converges to the graded space $\mathrm{Gr}H$. In our case we will be able to show that all differentials $d_n$ for $n > 2$ vanish and thus $E_3 = E_4 = \cdots = E_\infty$. Therefore, we will never have to calculate higher then the third terms in the spectral sequence. We will give some more details on the application of the method of spectral sequences to our case in Appendix A.

[2] For the $p = 0$ extended complex and only for this case we will be able to show that the cohomology can be represented by homogeneous polynomials, but this will come out as a non-trivial result.

# 3 Chiral extended complex

Before attempting a calculation of the cohomology of the full extended complex let us consider its chiral version. This is a warm up problem which, nevertheless, captures the major features.

We replace the Fock space $\mathcal{F}_p(\alpha, \overline{\alpha})$ by its chiral version, $\mathcal{F}_p(\alpha)$ which is generated from the vacuum by the left moving modes $\alpha_n$ only.

Repeating the arguments of the previous section we conclude that the chiral version of $\widehat{Q}$ is given by

$$\widehat{Q} = 1 \otimes Q - i\frac{\partial}{\partial x^\mu} \otimes \sum_n c_n \alpha^\mu_{-n} - \frac{1}{2}\eta^{\mu\nu}\Box \otimes c_0. \tag{3.1}$$

We will calculate the cohomology of chiral extended complex $\widehat{V}_p$ for three different cases: case $p^2 \neq 0$, which describes the massive spectrum; case $p^2 = 0$—the massless one; and case $p \equiv 0$, which besides the particular states from the massless spectrum describes a number of discrete states.

## 3.1 Massive states

Let us start the calculation of the cohomology of the chiral extended complex $\widehat{V}_p$ by considering the case of $p^2 \neq 0$. For this case the cohomology of the BRST complex is non-zero only for ghost number one and ghost number two. The cohomology contains the same number of ghost number one and two states which can be written in terms of dimension one primary matter states. Let $|v, p\rangle \in V_p$ be a dimension one primary state with no ghost excitations; then the following states,

$$c_0 c_1 |v, p\rangle \qquad \text{and} \qquad c_1 |v, p\rangle, \tag{3.1.1}$$

represent nontrivial cohomology classes and, moreover, each cohomology class has a representative of this kind (see ref. [10]).

We will calculate the cohomology of the extended complex in two steps. First, we extend the BRST complex by adding polynomials of one variable $\tilde{x} = (p \cdot x)$. The resulting space,

$$\widetilde{V}_p = \mathbb{C}[\tilde{x}] \otimes V_p, \tag{3.1.2}$$

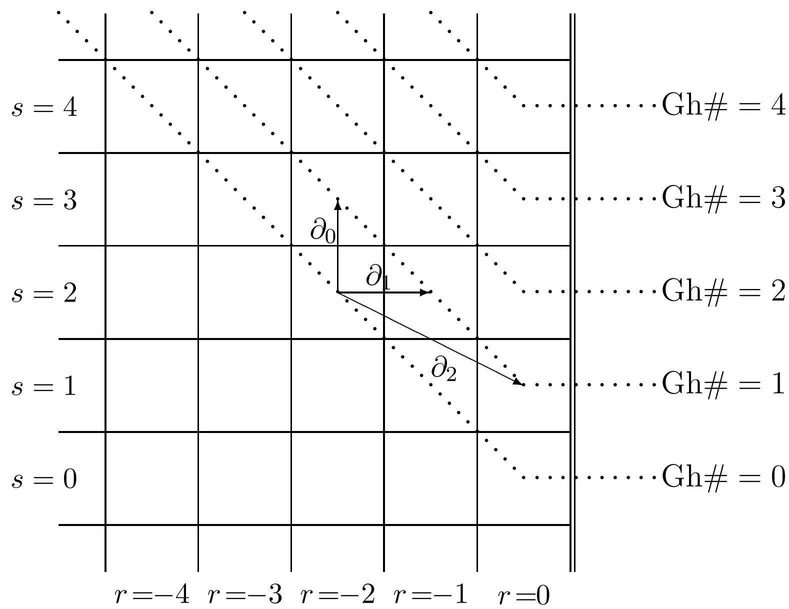


Figure 1: Anatomy of a double complex

is a subcomplex of $\widehat{V}_p$ and we define its cohomology as $H(\widetilde{Q},\widetilde{V}_p)$, where $\widetilde{Q}$ is the restriction of $\widehat{Q}$ on $\widetilde{V}_p$. Calculation of $\mathrm{Gr}H(\widetilde{Q},\widetilde{V}_p)$ is the objective of the first step. Second, we obtain the full extended space as a tensor product of $\widetilde{V}_p$ with the polynomials of the transverse variables

$$\widehat{V}_p=\mathbb{C}[\tilde{x}^1\ldots,\tilde{x}^{D-1}]\otimes\widetilde{V}_p, \tag{3.1.3}$$

where

$$\tilde{x}^i=x^i-\frac{p^i(p\cdot x)}{p^2}. \tag{3.1.4}$$

Using $\mathrm{Gr}H(\widetilde{Q},\widetilde{V}_p)$ found in the first step, we will calculate $\mathrm{Gr}H(\widehat{Q},\widehat{V}_p)$.

Let us calculate $\mathrm{Gr}H(\widetilde{Q},\widetilde{V}_p)$. Beside the ghost number, complex $\widetilde{V}_p$ has an additional grading—the $x$-degree. According to these two gradings we can write $\widetilde{V}_p$ as a double sum

$$\widetilde{V}_p=\bigoplus_{r,s}\widetilde{E}_0^{r,s}(p), \tag{3.1.5}$$

where $\widetilde{E}_0^{r,s}=\tilde{x}^{-r}\otimes V_p^{(r+s)}$ is the space of ghost number $r+s$ states with $-r$ factors of $\tilde{x}$. Note that in our notations $r\leq 0$.

It will be convenient to represent a double graded complex like $\widetilde{V}_p$ graphically by a lattice (see Fig. 1) where each cell represents a space $\widetilde{E}_0^{r,s}$, columns represent the spaces with definite $x$-degrees and the diagonals represent the spaces with definite ghost numbers.

The action of $\widetilde{Q}$ on $\widetilde{V}_p$ can be easily derived from the general formula (3.1). Any vector from $\widetilde{E}_0^{-k,s}$ can be represented as $\tilde{x}^k \otimes |v,p\rangle$, where $|v,p\rangle \in V_p^{(s-k)}$ is a vector from the BRST complex $V_p$ with ghost number $s-k$. Applying $\widetilde{Q}$ to this state we obtain

$$\begin{aligned} \widetilde{Q}\, \tilde{x}^k \otimes |v,p\rangle &= \tilde{x}^k \otimes Q|v,p\rangle \\ &- ik\, \tilde{x}^{k-1} \otimes \sum_n c_n\, (p\cdot\alpha_{-n})\, |v,p\rangle \\ &- \frac{k\,(k-1)}{2}\, p^2\, \tilde{x}^{k-2} \otimes c_0\, |v,p\rangle \end{aligned} \tag{3.1.6}$$

According to Eq. (3.1.6) we decompose $\widetilde{Q}$ in the sum of operators with a definite $x$-degree

$$\widetilde{Q} = \partial_0 + \partial_1 + \partial_2, \tag{3.1.7}$$

where each $\partial_n$ reduces the $x$-degree, or increases $r$, by $n$ (see Fig. 1).

Now we start building the spectral sequence of the complex $(\widetilde{V}_p, \widetilde{Q})$. For a short review of the method see Appendix A. The first step, the calculation of $\widetilde{E}_1^{r,s} = \mathrm{Gr}_r H^{r+s}(\partial_0, \widetilde{V})$, reduces to the calculation of the cohomology of the BRST complex. Indeed, according to Eq. (3.1.6), $\partial_0 = 1 \otimes Q$, and therefore

$$\widetilde{E}_1^{r,s} = \tilde{x}^{-r} \otimes H^{(r+s)}(Q, V_p). \tag{3.1.8}$$

As we mentioned above, the BRST complex has nontrivial cohomology only at ghost numbers one and two. Thus the space $\widetilde{E}_1 = \bigoplus \widetilde{E}_1^{r,s}$ looks as shown in Fig. 2 (left), where shaded cells correspond to non-zero spaces.

The differential $d_1$ is induced on $\widetilde{E}_1$ by $\partial_1$, and acts from $\widetilde{E}_1^{r,s}$ to $\widetilde{E}_1^{r+1,s}$ as shown in Fig. 2 (left). Since there are no states below the ghost number one and above ghost number two, the cohomology of $d_1$ at ghost number one is given by its kernel:

$$\widetilde{E}_2^{r,1-r} = \ker d_1, \tag{3.1.9}$$

and at ghost number two by the quotient of $\widetilde{E}_2^{r,2-r}$ by the image of $d_1$:

$$\widetilde{E}_1^{r,2-r} = \widetilde{E}_2^{r,2-r} / \operatorname{Im} d_1. \tag{3.1.10}$$

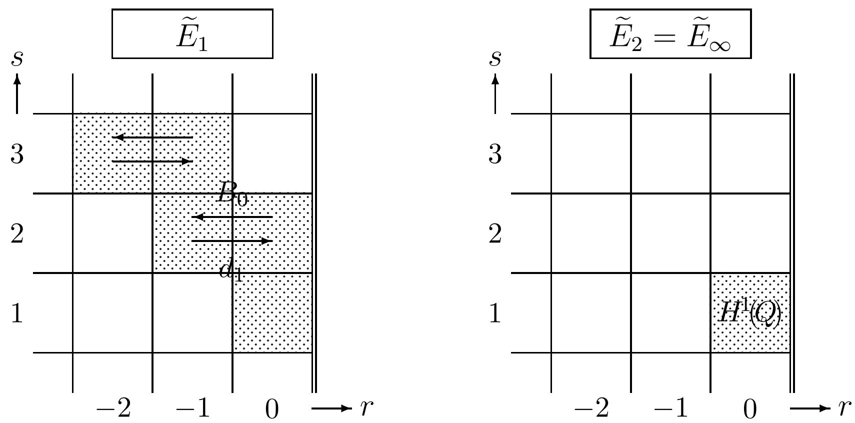


Figure 2: Spectral sequence for $\widetilde{V}_p$

We are going to show that $d_1$ establishes an isomorphism of the corresponding spaces and, therefore, the only non-empty component of $\widetilde{E}_2$ is $\widetilde{E}_2^{0,1} \simeq H^1(Q, V_p)$ as shown in Fig. 2 (right).

Consider an operator $B_0 = \tilde{x} \otimes b_0$. This operator is well defined on $\widetilde{E}_1$ *i.e.*, it maps cohomology classes to cohomology classes. On the other hand, its anticommutator with $d_1$ is given by

$$\{d_1, B_0\} = p^2\,\hat{k}, \tag{3.1.11}$$

where $\hat{k}$ the $x$-degree operator. The last equation shows that if $p^2 \neq 0$, non-trivial cohomology of $d_1$ may exist only in $\hat{k} = 0$ subspace of $\widetilde{E}_2$. Moreover, if we apply $\{d_1, B_0\} = d_1\,B_0 + B_0\,d_1$ to ghost number one states only the second term will survive because there are no ghost number zero states in $\widetilde{E}_1$. Thus we conclude that up to a diagonal matrix $B_0$ is an inverse operator to $d_1$. Since $\widetilde{E}_1^{r,1-r}$ and $\widetilde{E}_1^{r+1,1-r}$ have the same dimension and $d_1$ is invertible it is an isomorphism between $\widetilde{E}_2^{r,1-r}$ and $\widetilde{E}_2^{r+1,1-r}$ for any $r \leq 0$.

As shown in Fig. 2 (right), $\widetilde{E}_2$ contains only one non-empty component. This means that second differential $d_2$ and all higher are necessarily zero and the spectral sequence collapses at $\widetilde{E}_2 = \widetilde{E}_\infty$. Therefore, we conclude that

$$\mathrm{Gr}H(\widetilde{Q}, \widetilde{V}_p) = H^1(Q, V_p). \tag{3.1.12}$$

The second step in our program is trivial because the spectral sequence $\{\widehat{E}_n\}$ of the full complex

$$\widehat{V}_p = \mathbb{C}[\tilde{x}^1, \ldots, \tilde{x}^{D-1}] \otimes \widetilde{V}_p, \tag{3.1.13}$$

stabilizes at $\widehat{E}_1$ and

$$\widehat{E}_1 = \widehat{E}_\infty = \mathbb{C}[\tilde{x}_1, \ldots, \tilde{x}_{D-1}] \otimes \mathrm{Gr} H(\widetilde{Q}, \widetilde{V}_p). \tag{3.1.14}$$

This happens simply because, according to Eq. (3.1.12), $\mathrm{Gr} H(\widetilde{Q}, \widetilde{V}_p)$, and thus $\widehat{E}_1$, contains only ghost number one states and therefore $d_1$ and all higher differentials must vanish. Combining Eqs. (3.1.14) and (3.1.12) we obtain

$$\mathrm{Gr} H(\widehat{Q}, \widehat{V}_p) = \mathbb{C}[\tilde{x}^1, \ldots, \tilde{x}^{D-1}] \otimes H^1(Q, V_p). \tag{3.1.15}$$

This completes our analysis of the cohomology of the chiral extended complex for $p^2 \neq 0$.

## 3.2 Massless states

The analysis presented above can not be applied to the light-cone, $p^2 = 0$. We could, in principle, repeat all the arguments using $\xi \cdot x$ instead of $\tilde{x}$, where $\xi$ is some vector for which $\xi \cdot p \neq 0$, to build $\widetilde{V}$ and this would work everywhere except at the origin of the momentum space, $p = 0$. Yet it is instructive to make a covariant calculation in this case. Since there is no covariant way to choose a vector $\xi$ we can not apply our two step program. Instead we will start from scratch and build a spectral sequence for the whole module

$$\widehat{V}_p = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes V_p, \tag{3.2.1}$$

graded by the total $x$-degree.

According to Eq. (3.1), we can decompose $\widehat{Q}$ into a sum of operators of definite $x$-degree

$$\widehat{Q} = \partial_0 + \partial_1 + \partial_2, \tag{3.2.2}$$

where

$$\begin{aligned} \partial_0 &= 1 \otimes Q, \\ \partial_1 &= -i \frac{\partial}{\partial x^\mu} \otimes \sum_n c_n \alpha^\mu_{-n}, \\ \partial_2 &= -\frac{1}{2} \Box \otimes c_0. \end{aligned} \tag{3.2.3}$$

The first step is to find cohomology of $\partial_0$, which is just the tensor product of the BRST cohomology $H(Q, V_p)$ with the space of polynomials

$$E_1 = H(\partial_0, \widehat{V}_p) = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes H(Q, V_p). \tag{3.2.4}$$

Multiplying the representatives of $H(Q, V_p)$ by arbitrary polynomials in $x$ we obtain the following representatives of $E_1$ cohomology classes

$$\begin{aligned} &P_\mu(x) \otimes c_1 \alpha^\mu_{-1} |p\rangle, \\ &Q_\mu(x) \otimes c_0 c_1 \alpha^\mu_{-1} |p\rangle, \end{aligned} \tag{3.2.5}$$

where $P_\mu(x)$ and $Q_\mu(x)$ are polynomials in $x$ that satisfy the transversality condition, $p^\mu Q_\mu(x) = p^\mu P_\mu(x) = 0$, and are not proportional to $p_\mu$. These transversality conditions come from the same conditions on BRST cohomology classes at $p^2 = 0$. The first differential acts non-trivially from ghost number one to ghost number two states according to the following formula

$$d_1 : \quad P_\mu \to Q_\mu = -ip^\nu \frac{\partial}{\partial x^\nu} P_\mu. \tag{3.2.6}$$

It is easy to check that the map (3.2.6) is surjective and therefore $E_2^{r,s} = 0$ for $r + s = 2$. As expected the cohomology of the massless complex has a similar structure to that of the massive one. There are no cohomology states with ghost number two and there is an infinite tower of ghost number one states with different $x$-degree.

## 3.3 Cohomology of the zero momentum chiral complex

The zero momentum complex is exceptional. Already in the BRST cohomology we encounter additional "discrete" states at exotic ghost numbers (see ref. [11]). The cohomology is one dimensional at ghost numbers zero and three and $D$-dimensional at ghost numbers one and two. Explicit representatives for these classes can be written as given in Table 1.

Let us denote the direct sum of spaces $E_n^{r,s}$ with the same ghost number $m = r + s$ by $E_n^{(m)}$:

$$E_n^{(m)} \equiv \bigoplus_{r \leq 0} E_n^{r,m-r} \tag{3.3.1}$$

| Ghost # | Representatives | Dimension |
|---|---|---|
| 3 | $c_1 c_0 c_{-1}\|0\rangle$ | 1 |
| 2 | $c_0 \alpha^\mu_{-1}\|0\rangle$ | $D$ |
| 1 | $\alpha^\mu_{-1}\|0\rangle$ | $D$ |
| 0 | $\|0\rangle$ | 1 |

Table 1: Chiral BRST cohomology at $p = 0$

As usual, the cohomology of $\partial_0$ can be written in terms of the BRST cohomology as

$$E_1^{(m)} = H^m(\partial_0, \widehat{V}_0) = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes H^m(Q, V_0).$$

According to Table 1, $H^0(Q, V_0) \simeq H^3(Q, V_0) \simeq \mathbb{C}$. Therefore, $E_1^{(0)}$ and $E_1^{(3)}$ are isomorphic to the space of polynomials $\mathbb{C}[x^\mu]$. Similarly, $E_1^{(1)}$ and $E_1^{(2)}$ are isomorphic to the space of polynomial vector fields $\mathbb{C}^D[x^\mu]$. Evaluating the action of $\partial_1$ on the representatives (see Table 1), we get the following sequence

$$0 \to \mathbb{C}[x^\mu] \xrightarrow{\nabla} \mathbb{C}^D[x^\mu] \xrightarrow{0} \mathbb{C}^D[x^\mu] \xrightarrow{\nabla} \mathbb{C}[x^\mu] \to 0, \tag{3.3.2}$$

where the first nabla-operator is the gradient, which maps scalars to vectors and the second is the divergence, which does the opposite (and we have dropped an insignificant factor $-i$). It is convenient to interpret $\mathbb{C}[x^\mu] = \Omega^0$ and $\mathbb{C}^D[x^\mu] = \Omega^1$ as a space of polynomial zero and one-forms on the Minkowski space. The first nabla-operator in Eq. (3.3.2) will be interpreted as an exterior derivative $d$ and the second as its Hodge conjugate $\delta$. With new notation we rewrite the sequence in Eq. (3.3.2) as

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{0} \Omega^1 \xrightarrow{\delta} \Omega^0 \to 0. \tag{3.3.3}$$

This is illustrated by Fig. 3 (left). Note that individual cells in Fig. 3 correspond to the subspaces of homogeneous polynomials rather than whole $\Omega^1$

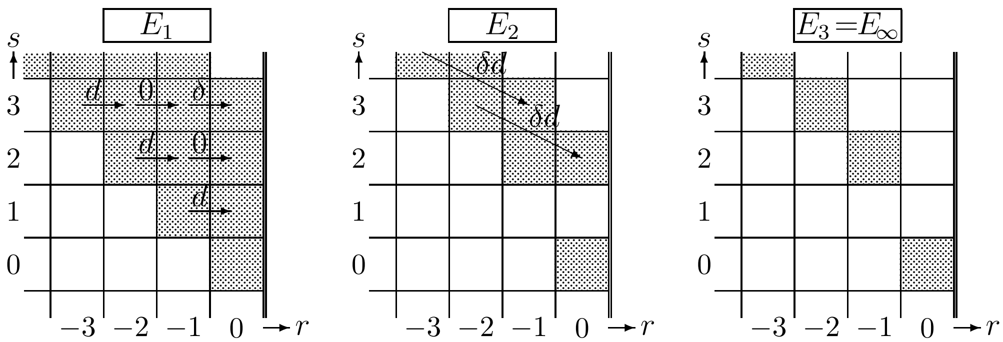


Figure 3: Spectral sequence for $\widehat{V}_0$

or $\Omega^0$. One can easily calculate cohomology of this complex and obtain

$$\begin{aligned} E_2^{(0)} &= \mathbb{C}, \\ E_2^{(1)} &= \Omega^1/d\Omega^0, \\ E_2^{(2)} &= \ker(\delta, \Omega^1), \\ E_2^{(3)} &= 0. \end{aligned} \tag{3.3.4}$$

Thus the second term in the spectral sequence has the structure presented by Fig. 3 (middle).

Now we have to calculate $d_2$. This differential acts as shown in Fig. 3 (middle) and can be found from the following formula (see Appendix A):

$$d_2 = \partial_2 - \partial_1 \partial_0^{-1} \partial_1. \tag{3.3.5}$$

Let $P = P_\mu dx^\mu \in \Omega^1$ be a polynomial one form. Corresponding state representing a $E_2^{(1)}$ cohomology class is given by

$$P_\mu \otimes c_1 \alpha_{-1}^\mu |0\rangle. \tag{3.3.6}$$

Applying $\partial_1$ to (3.3.6) we obtain

$$\partial_1 \; P_\mu \otimes \alpha_{-1}^\mu c_1 |0\rangle = -i \frac{\partial P_\mu}{\partial x_\mu} \otimes c_{-1} c_1 |0\rangle = \partial_0 \; i \frac{1}{2} \frac{\partial P_\mu}{\partial x_\mu} \otimes c_0 |0\rangle, \tag{3.3.7}$$

where we use the metric $\eta^{\mu\nu}$ to raise and lower indices. From Eq. (3.3.7) we derive that

$$\partial_1 \partial_0^{-1} \partial_1 \; P_\mu \otimes \alpha_{-1}^\mu c_1 |0\rangle = \partial_1 \, \frac{i}{2} \frac{\partial P_\mu}{\partial x_\mu} \otimes c_0 |0\rangle = \frac{1}{2} \frac{\partial^2 P_\mu}{\partial x_\mu \partial x^\nu} \otimes \alpha_{-1}^\nu c_1 c_0 |0\rangle. \quad (3.3.8)$$

With the definition of $\partial_2$ (see Eq. (3.2.3)) we immediately get

$$\partial_2 \; P_\mu \otimes \alpha_{-1}^\mu c_1 |0\rangle = \frac{1}{2} \Box P_\mu c_1 c_0 |0\rangle. \quad (3.3.9)$$

By adding the last two equations, and dropping an insignificant factor 1/2, we see that the second differential acts on $E_2^{(1)} = \mathbb{C}^D[x^\mu]/\nabla\mathbb{C}[x^\mu]$ as the following matrix differential operator:

$$(d_2)_\mu^\nu = \Box \delta_\mu^\nu - \partial_\mu \partial^\nu, \quad (3.3.10)$$

or, using the differential form interpretation $E_2^{(1)} = \Omega^1 / d\Omega^0$,

$$d_2 = \Box - d\delta = \delta d. \quad (3.3.11)$$

In order to calculate $E_3 = H(d_2, E_2)$ we have to calculate the cohomology of the following complex:

$$0 \to \mathbb{C} \xrightarrow{0} \Omega^1 / d\Omega^0 \xrightarrow{\delta d} \ker(\delta, \Omega^1) \to 0, \quad (3.3.12)$$

or equivalently

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{\delta d} \Omega^1 \xrightarrow{\delta} \Omega^0 \to 0. \quad (3.3.13)$$

The later sequence is known to be exact in the last two terms[3] and thus $E_3$ is given by

$$\begin{aligned} E_3^{(0)} &= \mathbb{C}, \\ E_3^{(1)} &= \ker(\delta d, \Omega^1)/d\Omega^0, \\ E_3^{(2)} &= E_3^{(3)} \; = \; 0. \end{aligned} \quad (3.3.14)$$

Looking at Fig. 3 (right) one can easily deduce that all differentials $d_k$ for $k \geq 3$ vanish. This allows us to conclude that

$$\mathrm{Gr}_r H^{r+s}(\widehat{Q}, \widehat{V}_0) = E_\infty^{r,s} = E_3^{r,s}, \quad (3.3.15)$$

[3] We would like to thank Jeffrey Goldstone for pointing this out to us.

and Gr$H$ has the structure shown in Fig. 3 (right). Since cohomology is non-trivial only at one ghost number for every $x$-degree, we can easily find the dimensions of different Gr$_r H$ spaces. We present the summary of our results on the cohomology of the chiral extended complex in the following theorem.

**Theorem 1** *The cohomology of the chiral extended complex $\widehat{V}_p$ admits a natural filtration (by the* minus *$x$-degree) $\mathcal{F}_r H(\widehat{V}_p, \widehat{Q})$. The cohomology can be described using the associated graded spaces*

$$\mathrm{Gr}_r H(\widehat{V}_p, \widehat{Q}) = \mathcal{F}_r H(\widehat{V}_p, \widehat{Q}) / \mathcal{F}_{r+1} H(\widehat{V}_p, \widehat{Q})$$

*as follows*

1. *Non-zero momentum ($p \not\equiv 0$): cohomology is trivial unless $p^2 = n - 1$, where $n$ is a non-negative integer. In the latter case*

$$\dim \mathrm{Gr}_r H^1(\widehat{V}_p, \widehat{Q}) = \binom{24 - r}{24} d_n,$$
$$\dim H^l(\widehat{V}_p, \widehat{Q}) = 0 \qquad \textit{for } l \neq 1,$$

*where $d_n$ denotes the number of the BRST states at mass level $n$. These numbers are generated by*

$$\prod_{k=1}^{\infty} (1 - z^k)^{-24} = \sum_{n=1}^{\infty} d_n z^n.$$

*see ref. [2].*

2. *Zero momentum ($p \equiv 0$): the cohomology appears at ghost number zero (the vacuum)*

$$\dim H^0(\widehat{V}_p, \widehat{Q}) = 1,$$

*and ghost number one (physical states, recall that $r$ is* minus *$x$-degree).*

$$\dim \mathrm{Gr}_r H^1(\widehat{V}_p, \widehat{Q}) = \begin{cases} 0 & \textit{if } r = 0; \\ \dfrac{D(D-1)}{2} & \textit{for } r = -1; \\ \dfrac{D(D-2)(D+2)}{3} & \textit{for } r = -2; \\ \chi_r & \textit{for } r \leq -3. \end{cases}$$

*where*

$$\chi_r = D\binom{D-1-r}{D-1} + \binom{D-4-r}{D-1} - \binom{D-r}{D-1} - D\binom{D-3-r}{D-1}.$$

*The cohomology is trivial for all the other ghost numbers*

$$\dim H^l(\widehat{V}_p, \widehat{Q}) = 0 \qquad \textit{for } l \neq 0, 1.$$

# 4 Cohomology of full extended complex

In this section we will calculate the semi-relative cohomology of the full extended complex. We will use the same technique as for the chiral complex and obtain similar results.

## 4.1 Review of semi-relative BRST cohomology

By definition, the semi-relative cohomology $\mathcal{H}_S$ consists of $Q$ invariant states annihilated by $b_0 - \overline{b}_0$, modulo $Q|\Lambda\rangle$ where $|\Lambda\rangle$ is annihilated by $b_0 - \overline{b}_0$. For future convenience we set $b_0^{\pm} = b_0 \pm \overline{b}_0$ and $c_0^{\pm} = (1/2)(c_0 \pm \overline{c}_0)$.

Semi-relative cohomology $\mathcal{H}_S$ of the BRST complex can be easily expressed in terms of relative cohomology $\mathcal{H}_R$. The latter consists of the $Q$ invariant states annihilated both by $b_0$ and $\overline{b}_0$ modulo $Q|\Lambda\rangle$ where $|\Lambda\rangle$ is also annihilated by $b_0$ and $\overline{b}_0$ separately. We can express $\mathcal{H}_S$ in terms of $\mathcal{H}_R$ using the following exact sequence (see ref. [12]):

$$\cdots \to \mathcal{H}_R^n \xrightarrow{i} \mathcal{H}_S^n \xrightarrow{b_0^+} \mathcal{H}_R^{n-1} \xrightarrow{\{Q,c_0^+\}} \mathcal{H}_R^{n+1} \xrightarrow{i} \mathcal{H}_S^{n+1} \to \cdots. \qquad (4.1.1)$$

The map $i$ is induced in the cohomology by the natural embedding of the relative complex into the absolute complex. The relative cohomology can be found as a tensor product of the relative cohomologies of the left and right chiral sectors.

This is particularly easy for the case $p \neq 0$ because in this case the chiral relative cohomology is non-zero only at ghost number one (see [10]) and, therefore, $\mathcal{H}_R$ consists of ghost number two states only. The exact sequence (4.1.1) reduces in this case to

$$0 \to \mathcal{H}_S^3 \xrightarrow{b_0^+} \mathcal{H}_R^2 \to 0 \qquad \text{and} \qquad 0 \to \mathcal{H}_R^2 \xrightarrow{i} \mathcal{H}_S^2 \to 0, \qquad (4.1.2)$$

which means that $\mathcal{H}_S^3$ is isomorphic to $\mathcal{H}_S^2$ which in turn is isomorphic to $\mathcal{H}_R^2$ and there are no semi-relative cohomology states at ghost numbers other than two or three. We can even write explicit representatives in terms of dimension $(1,1)$ primary matter states. Let $|v,p\rangle$ denote such a state. Representatives of semi-relative cohomology classes can be written as

$$c_1\overline{c}_1|v,p\rangle \qquad \text{and} \qquad c_0^+c_1\overline{c}_1|v,p\rangle. \tag{4.1.3}$$

## 4.2 Semi-relative cohomology of $p \not\equiv 0$ extended complex

The calculation of the extended cohomology for the semi-relative complex in the case of $p \not\equiv 0$ is line by line parallel to that of the chiral one. When we add polynomials of $\tilde{x} = (p \cdot x)$ the resulting cohomology is given by ghost number two BRST states only, and the semi-relative cohomology of the full extended complex is obtained by adding polynomials of transverse components of $x$:

$$\mathrm{Gr}\,\mathcal{H}_S(\widehat{Q}, \widehat{V}_p) = \mathbb{C}[\tilde{x}^1, \cdots, \tilde{x}^{D-1}] \otimes \mathcal{H}_S^2. \tag{4.2.4}$$

The case of the massless ($p^2 = 0$) complex may require some special consideration because there is no straightforward way to choose the transverse variables $\tilde{x}^i$ but the answer is the same.

In our opinion, an important result is that the extended cohomology does not contain ghost number three states. Let us explain in more details what happens to the ghost number three semi-relative states when we extend the complex by $x_0$. According to the results on extended cohomology for every state $c_0^+c_1\overline{c}_1|v,p\rangle$ we should be able to find a vector $|w,p\rangle$ in the extended complex such that $c_0^+c_1\overline{c}_1|v,p\rangle = \widehat{Q}|w,p\rangle$. We have found the leading part of $|w,p\rangle$ when we calculated the spectral sequence. It is given by[4] $\frac{(p\cdot x)}{p^2}c_1\overline{c}_1|v,p\rangle$. Applying $\widehat{Q}$ to this state we get

$$\widehat{Q}\frac{(p\cdot x)}{p^2}c_1\overline{c}_1|v,p\rangle = c_0^+c_1\overline{c}_1|v,p\rangle + \sum_n p_\mu c_1\overline{c}_1(c_{-n}\alpha_n^\mu + \overline{c}_{-n}\overline{\alpha}_n^\mu)|v,p\rangle. \tag{4.2.5}$$

Thus in order to prove that $c_0^+c_1\overline{c}_1|v,p\rangle$ is trivial we have to prove that the sum in the right hand side represents a $\widehat{Q}$ exact state. The latter is a direct

[4]This is not applicable to the massless case $p^2 = 0$ which has to be analyzed separately.

consequence of the absence of relative cohomology of ghost number three. Indeed, this sum is annihilated by $b_0$, $\overline{b}_0$, and $Q$ and if it were nontrivial it would have represented a non-zero cohomology class of ghost number three in the relative complex.

## 4.3 Semi-relative cohomology of $p = 0$ extended complex

The calculation of the cohomology of the zero-momentum extended complex is based on the same ideas that we used for the chiral case but is technically more difficult. The major complication is due to a much larger BRST cohomology.

We can find the BRST cohomology of the zero momentum semi-relative complex using the long exact sequence we mentioned above (see Eq. (4.1.1)) which in this case breaks in to the following exact sequences:

$$\begin{aligned} 0 \to \mathcal{H}_R^0 &\xrightarrow{i} \mathcal{H}_S^0 \xrightarrow{b_0^+} 0, \\ 0 \to \mathcal{H}_R^1 &\xrightarrow{i} \mathcal{H}_S^2 \xrightarrow{b_0^+} \mathcal{H}_R^0 \xrightarrow{\{Q,c_0^+\}} \mathcal{H}_R^2, \\ 0 \to \mathcal{H}_R^1 &\xrightarrow{i} \mathcal{H}_S^3 \xrightarrow{b_0^+} \mathcal{H}_R^0 \xrightarrow{\{Q,c_0^+\}} \mathcal{H}_R^2, \\ \mathcal{H}_R^2 \xrightarrow{\{Q,c_0^+\}} \mathcal{H}_R^4 &\xrightarrow{i} \mathcal{H}_S^4 \xrightarrow{b_0^+} \mathcal{H}_R^3 \to 0, \\ 0 \to \mathcal{H}_S^5 &\xrightarrow{b_0^+} \mathcal{H}_R^4 \to 0. \end{aligned} \tag{4.3.1}$$

The cohomology of the relative complex can be found as a tensor product of the left and right relative cohomologies. The later can be found, for example, in ref. [10] and consists of ghost number zero, one, and two states. Therefore, the relative complex has non-trivial cohomologies at ghost numbers from zero through four and their representatives and dimensions are listed in Table 2.

Using the exact sequences (4.3.1) we can easily find the dimensions and representatives of the semi-relative cohomology as shown in Table 3.

The extended module $\widehat{V}_0$ is given by

$$\widehat{V}_0 = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes V_0, \tag{4.3.2}$$

and the operator $\widehat{Q}$ has the following $x$-degree decomposition

$$\widehat{Q} = \partial_0 + \partial_1 + \partial_2, \tag{4.3.3}$$

| Ghost # | Representatives | Dimension |
|---|---|---|
| 0 | $\|0\rangle$ | 1 |
| 1 | $c_1\alpha^\mu_{-1}\|0\rangle, \quad \overline{c}_1\overline{\alpha}^\mu_{-1}\|0\rangle$ | $2D$ |
| 2 | $c_1\alpha^\mu_{-1}\overline{c}_1\overline{\alpha}^\nu_{-1}\|0\rangle, \quad c_1c_{-1}\|0\rangle, \quad \overline{c}_1\overline{c}_{-1})\|0\rangle$ | $D^2+2$ |
| 3 | $c_1\alpha^\mu_{-1}\overline{c}_1\overline{c}_{-1}\|0\rangle, \quad c_1c_{-1}\overline{c}_1\overline{\alpha}^\nu_{-1}\|0\rangle$ | $2D$ |
| 4 | $c_1c_{-1}\overline{c}_1\overline{c}_{-1}\|0\rangle$ | 1 |

Table 2: Cohomology of the relative BRST complex at $p = 0$

where

$$\begin{aligned} \partial_0 &= 1 \otimes Q, \\ \partial_1 &= -i\frac{\partial}{\partial x^\mu} \otimes \sum_n (c_n\alpha^\mu_{-n} + \overline{c}_n\overline{\alpha}^\mu_{-n}), \\ \partial_2 &= -\frac{1}{2}\Box \otimes (c_0 + \overline{c}_0) = -\Box \otimes c^+. \end{aligned} \tag{4.3.4}$$

As it was the case for the chiral complex, the cohomology of $\partial_0$ coincides with the cohomology of the semirelative BRST module, $\mathcal{H}_S$ tensored with $\mathbb{C}[x^0, \ldots, x^{D-1}]$ and the first term in the spectral sequence is given by

$$E_1 = H_S(\partial_0, \widehat{V}_0) = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes \mathcal{H}_S \tag{4.3.5}$$

Using information from Table 3, we can parameterize the space $E_1 = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes \mathcal{H}_S$ as follows

$$\begin{aligned} (R^{[0]}) &= R^{[0]} \otimes |0\rangle, \\ (P^{[1]}_\mu,\ \overline{P}^{[1]}_\mu) &= P^{[1]}_\mu \otimes c_1\alpha^\mu_{-1}|0\rangle + \overline{P}^{[1]}_\mu \otimes \overline{c}_1\overline{\alpha}^\mu_{-1}|0\rangle, \\ (Q^{[2]}_{\mu\nu},\ R^{[2]}) &= Q^{[2]}_{\mu\nu} \otimes c_1\alpha^\mu_{-1}\overline{c}_1\overline{\alpha}^\nu_{-1}|0\rangle + R^{[2]} \otimes (c_1c_{-1} - \overline{c}_1\overline{c}_{-1})|0\rangle, \\ (Q^{[3]}_{\mu\nu},\ R^{[3]}) &= Q^{[3]}_{\mu\nu} \otimes c_0^+c_1\alpha^\mu_{-1}\overline{c}_1\overline{\alpha}^\nu_{-1}|0\rangle + R^{[3]} \otimes c_0^+(c_1c_{-1} - \overline{c}_1\overline{c}_{-1})|0\rangle, \\ (P^{[4]}_\mu,\ \overline{P}^{[4]}_\mu) &= P^{[4]}_\mu \otimes c_0^+c_1\alpha^\mu_{-1}\overline{c}_1\overline{c}_{-1}|0\rangle + \overline{P}^{[4]}_\mu \otimes c_0^+c_1c_{-1}\overline{c}_1\overline{\alpha}^\mu_{-1}|0\rangle, \\ (R^{[5]}) &= R^{[0]} \otimes c_0^+c_1c_{-1}\overline{c}_1\overline{c}_{-1}|0\rangle, \end{aligned} \tag{4.3.6}$$

| Ghost # | Representatives | Dimension |
|---|---|---|
| 0 | $\vert 0\rangle$ | 1 |
| 1 | $c_1\alpha^{\mu}_{-1}\vert 0\rangle, \quad \overline{c}_1\overline{\alpha}^{\mu}_{-1}\vert 0\rangle$ | $2D$ |
| 2 | $c_1\alpha^{\mu}_{-1}\overline{c}_1\overline{\alpha}^{\nu}_{-1}\vert 0\rangle, \quad (c_1c_{-1} - \overline{c}_1\overline{c}_{-1})\vert 0\rangle$ | $D^2+1$ |
| 3 | $c_0^+c_1\alpha^{\mu}_{-1}\overline{c}_1\overline{\alpha}^{\nu}_{-1}\vert 0\rangle, \quad c_0^+(c_1c_{-1} - \overline{c}_1\overline{c}_{-1})\vert 0\rangle$ | $D^2+1$ |
| 4 | $c_0^+c_1\alpha^{\mu}_{-1}\overline{c}_1\overline{c}_{-1}\vert 0\rangle, \quad c_0^+c_1c_{-1}\overline{c}_1\overline{\alpha}^{\mu}_{-1}\vert 0\rangle$ | $2D$ |
| 5 | $c_0^+c_1c_{-1}\overline{c}_1\overline{c}_{-1}\vert 0\rangle$ | 1 |

Table 3: BRST cohomology of the semi-relative complex at $p=0$

where $Q^{\mu\nu}$, $P_\mu$ and $\overline{P}_\mu$ and $R$ are correspondingly tensor-, vector-, and scalar-valued polynomials of $D$ variables $x^\mu$. Applying $\partial_1$ to the representatives given by (4.3.6) and dropping $Q$-trivial states we can find that the first differential acts according to the following diagram

$$
\begin{array}{lll}
0: & (R^{[0]}) \overset{d_1}{\searrow} & (0) \\
1: & (P^{[1]}_\mu,\ \overline{P}^{[1]}_\mu) \overset{d_1}{\searrow} & (\partial_\mu R^{[0]},\ \partial_\mu R^{[0]}) \\
2: & (Q^{[2]}_{\mu\nu},\ R^{[2]}) \overset{d_1}{\searrow} & (\partial_\mu \overline{P}^{[1]}_\nu - \partial_\nu P^{[1]}_\mu,\ \partial^\mu \overline{P}^{[1]}_\mu - \partial^\mu P^{[1]}_\mu) \\
3: & (Q^{[3]}_{\mu\nu},\ R^{[3]}) \overset{d_1}{\searrow} & (0,\ 0) \\
4: & (P^{[4]}_\mu,\ \overline{P}^{[4]}_\mu) \overset{d_1}{\searrow} & (\partial^\nu Q^{[3]}_{\mu\nu} + \partial_\mu R^{[3]},\ \partial^\nu Q^{[3]}_{\nu\mu} + \partial_\mu R^{[3]}) \\
5: & (R^{[5]}) & (\partial_\mu P^{[4]} - \partial_\mu \overline{P}^{[4]})
\end{array}
\tag{4.3.7}
$$

Consider the cohomology of $d_1$. From the explicit formulae (4.3.7) it is clear that $E_2 = H(d_1, E_1)$ contains no ghost number five states and the only state that survives at ghost number zero is given by a constant $R^{[0]}$ and is the vacuum. It is less trivial to show that $E_2^{(4)} = 0$, and that $E_2^{(1)}$ contains only $x$-degree one and two states. We will prove these two results in Appendix B

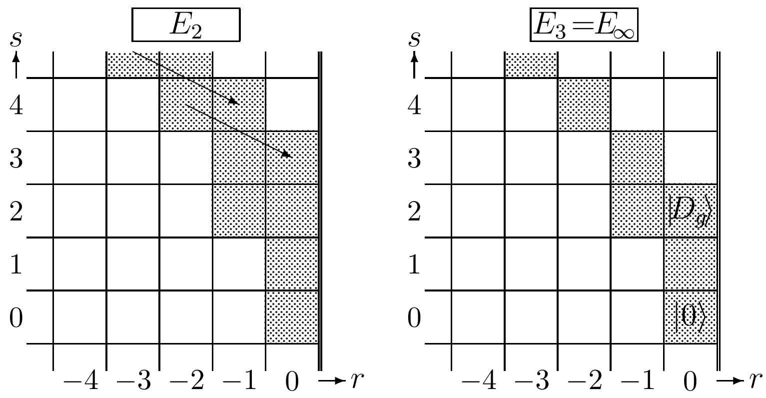


Figure 4: Spectral sequence for the zero momentum closed string states.

(lemmas 3 and 1). This shows that the space $E_2$ looks as shown in Fig. 4 (left). From the structure of $E_2$ we conclude that $d_2$ may only act from $E_2^{(2)}$ to $E_2^{(3)}$. In order to find

$$d_2\ (Q^{[2]}_{\mu\nu},\ R^{[2]}) = (\partial_2 - \partial_1\partial_0^{-1}\partial_1)\ (Q^{[2]}_{\mu\nu},\ R^{[2]}),$$

we have to keep the $\partial_0$ trivial terms when we apply $\partial_1$ to $(Q^{[2]}_{\mu\nu},\ R^{[2]})$ which were dropped in Eq. (4.3.7). Using (4.3.6) and (4.3.4) we infer

$$\begin{aligned}\partial_1\ (Q^{[2]}_{\mu\nu},\ R^{[2]}) = \partial_0\Big(&(\partial^\nu Q^{[2]}_{\mu\nu} + \partial_\mu R^{[2]})\otimes c_0^+\overline{c}_1\overline{\alpha}^\mu_{-1}|0\rangle\\ &-(\partial^\mu Q^{[2]}_{\mu\nu} + \partial_\nu R^{[2]})\otimes c_0^+ c_1\alpha^\nu_{-1}|0\rangle\Big)\,.\end{aligned}\tag{4.3.8}$$

Now we can apply $\partial_1$ to the argument of $\partial_0$ above and add the image of $\partial_2$ to obtain

$$d_2(Q^{[2]}_{\mu\nu},\ R^{[2]}) = (\delta Q^{[3]}_{\mu\nu},\ \delta R^{[3]}),\tag{4.3.9}$$

where

$$\begin{aligned}\delta Q^{[3]}_{\mu\nu} &= \Box Q^{[2]}_{\mu\nu} - \partial^\lambda\partial_\mu Q^{[2]}_{\lambda\nu} - \partial^\lambda\partial_\nu Q^{[2]}_{\mu\lambda} + 2\,\partial_\mu\partial_\nu R^{[2]},\\ \delta R^{[3]} &= -\partial^\lambda\partial^\rho Q^{[2]}_{\lambda\rho} + 2\,\Box R^{[2]}.\end{aligned}\tag{4.3.10}$$

As we will prove in Appendix B, this map is surjective and therefore the cohomology at ghost number three is trivial. The structure of $E_3$ is presented

in Fig. 4 (right). It is obvious that the spectral sequence collapses at $E_3$ and $E_\infty = E_3$. Therefore, we obtain the result that is similar to that in the chiral case (see Eq. (3.3.15)):

$$\mathrm{Gr}_r H^{r+s}(\widehat{Q}, \widehat{V}_0) = E_\infty^{r,s} = E_3^{r,s}. \tag{4.3.11}$$

Note that $d_2$ acts non-trivially only between ghost number two and ghost number three states, at the point where $d_1$ vanishes. This observation allows us to combine the calculation of $E_2$ and $E_3$ into one cohomology problem for the following complex

$$0 \to V^{(0)} \xrightarrow{d_1} V^{(1)} \xrightarrow{d_1} V^{(2)} \xrightarrow{d_2} V^{(3)} \xrightarrow{d_1} V^{(4)} \xrightarrow{d_1} V^{(5)} \to 0, \tag{4.3.12}$$

where $V^{(k)} = \bigoplus_r E_1^{r,k-r}$ are the ghost number $k$ subspaces of $E_1$. This is very similar to what we did for the chiral complex (Eq. (3.3.12) and Eq. (3.3.13)), but, unfortunately, we lack a simple geometrical interpretation for the complex above.

Since $d_1$ and $d_2$ have $x$-degree $-1$ and $-2$ correspondingly, we can decompose the complex above into the sum of the following subcomplexes

$$0 \to V_{n+2}^{(0)} \xrightarrow{d_1} V_{n+1}^{(1)} \xrightarrow{d_1} V_n^{(2)} \xrightarrow{d_2} V_{n-2}^{(3)} \xrightarrow{d_1} V_{n-3}^{(4)} \xrightarrow{d_1} V_{n-4}^{(5)} \to 0, \tag{4.3.13}$$

where the subscripts refer to $x$-degree. Using this decomposition and the results of Appendix B one can easily calculate the dimensions of $\mathrm{Gr}_r H_S$ spaces.

Let us summarize our results on the semi-relative cohomology of the extended complex.

**Theorem 2** *The semi-relative cohomology of the extended complex $\widehat{V}_p$, admits a natural filtration (by the* minus *$x$-degree) $\mathcal{F}_r H_S(\widehat{V}_p, \widehat{Q})$. The cohomology can be described using the associated graded spaces*

$$\mathrm{Gr}_r H_S(\widehat{V}_p, \widehat{Q}) = \mathcal{F}_r H_S(\widehat{V}_p, \widehat{Q}) / \mathcal{F}_{r+1} H_S(\widehat{V}_p, \widehat{Q})$$

*as follows*

1. *Non-zero momentum ($p \not\equiv 0$): cohomology is trivial unless $p^2 = 2n-2$, where $n$ is a non-negative integer. In the latter case*

$$\dim \mathrm{Gr}_r H_S^2(\widehat{V}_p, \widehat{Q}) = \binom{24-r}{24} d_n^2,$$
$$\dim H_S^l(\widehat{V}_p, \widehat{Q}) = 0 \qquad \text{for } l \neq 2$$

*where $d_n$ are generated by the following partition function*

$$\prod_{k=1}^{\infty}(1-z^k)^{-24} = \sum_{n=1}^{\infty} d_n z^n .$$

*see ref. [2].*

2. *Zero momentum case ($p \equiv 0$): the cohomology appears at ghost number zero (the vacuum)*

$$\dim H^0_S(\widehat{V}_p, \widehat{Q}) = 1,$$

*ghost number one (global symmetries of the background, the Poincaré algebra),*

$$\dim \mathrm{Gr}_r H^1_S(\widehat{V}_p, \widehat{Q}) = \begin{cases} D & \textit{if } r = 0; \\ \dfrac{D(D-1)}{2} & \textit{if } r = -1; \\ 0 & \textit{otherwise}; \end{cases}$$

*and ghost number two (physical states),*

$$\begin{aligned}
\dim \mathrm{Gr}_0 H^2_S(\widehat{V}_p, \widehat{Q}) &= 1, \\
\dim \mathrm{Gr}_{-1} H^2_S(\widehat{V}_p, \widehat{Q}) &= \frac{D\,(D^2-3\,D+8)}{6}, \\
\dim \mathrm{Gr}_{-2} H^2_S(\widehat{V}_p, \widehat{Q}) &= \frac{D(D-2)(D+2)}{3}, \\
\dim \mathrm{Gr}_{-3} H^2_S(\widehat{V}_p, \widehat{Q}) &= \frac{(D+2)\,(5\,D^3-16\,D^2+15\,D-12)}{24}, \\
\dim \mathrm{Gr}_r H^2_S(\widehat{V}_p, \widehat{Q}) &= \chi_r \ \textit{for } r \leq -4.
\end{aligned}$$

*where*

$$\begin{aligned}
\chi_r = (D^2+1) & \left[\binom{D-1-r}{D-1} - \binom{D-3-r}{D-1}\right] \\
& +2D\left[\binom{D-r-4}{D-1} - \binom{D-r}{D-1}\right] \\
& +\binom{D-r+1}{D-1} - \binom{D-r-5}{D-1}.
\end{aligned}$$

*There are no non-trivial cohomology states at any other ghost number,*

$$\dim H^l_S(\widehat{V}_p, \widehat{Q}) = 0 \qquad \textit{for } l \neq 0, 1, 2.$$

There is no evident structure among the ghost number two cohomology states. In the next section we are going to make some order in this zoo of zero momentum physical states using their transformation properties under the Lorentz group.

# 5 Lorentz group and extended complex

Since the cohomology of the extended complex is designed to describe the string theory near the flat background, the Lorentz group should act on it naturally, mapping the cohomology states to the cohomology states. In this section we investigate the action of the generators $J^{\mu\nu}$ of the infinitesimal Lorentz transformations.

## 5.1 Lorentz group acting on non-zero momentum extended complex

The generators of the infinitesimal Lorentz transformations, or the angular momentum operators of the closed string can be written as [2]

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + \overline{E}^{\mu\nu}, \tag{5.1.1}$$

where

$$\begin{aligned} l^{\mu\nu} &= x_0^\mu \alpha_0^\nu - x_0^\nu \alpha_0^\mu, \\ E^{\mu\nu} &= -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n), \\ \overline{E}^{\mu\nu} &= -i \sum_{n=1}^{\infty} \frac{1}{n} (\overline{\alpha}^\mu_{-n} \overline{\alpha}^\nu_n - \overline{\alpha}^\nu_{-n} \overline{\alpha}^\mu_n). \end{aligned} \tag{5.1.2}$$

Applying the zero mode part $l^{\mu\nu}$ to $\widehat{V}_p$, we obtain

$$l^{\mu\nu} = x^\mu \otimes p^\nu - x^\nu \otimes p^\mu + x^\mu \frac{\partial}{\partial x_\nu} \otimes 1 - x^\nu \frac{\partial}{\partial x_\mu} \otimes 1. \tag{5.1.3}$$

One can easily check that the $J^{\mu\nu}$ operators commute with $\widehat{Q}$ and therefore are well defined on the cohomology. Note that for $p \not\equiv 0$ these operators mix states of different $x$-degree.

Consider the massive case, $p^2 \not\equiv 0$. Recall that any physical state in the extended complex can be represented by

$$P(\tilde{x}^1, \cdots, \tilde{x}^{D-1}) \otimes c_1\overline{c}_1|v,p\rangle + \cdots , \tag{5.1.4}$$

where $|v,p\rangle$ is a dimension $(1,1)$ primary matter state with the momentum $p$ and dots stand for the lower $x$-degree terms. Suppose the transverse components are chosen as

$$\tilde{x}^i = x^i - \frac{p^i(p\cdot x)}{p^2}. \tag{5.1.5}$$

We can obtain a state with the same leading term by applying the following operator to $c_1\overline{c}_1|v,p\rangle$

$$P\left(\frac{p_\mu J^{1\mu}}{p^2}, \cdots, \frac{p_\mu J^{D-1\mu}}{p^2}\right)\ c_1\overline{c}_1|v,p\rangle = P(\tilde{x}^1, \cdots, \tilde{x}^{D-1}) \otimes c_1\overline{c}_1|v,p\rangle + \cdots . \tag{5.1.6}$$

Together with the results on the cohomology of the extended complex obtained in the previous section, Eq. (4.2.4), this shows that the semi-relative cohomology of a $p^2 \neq 0$ extended complex is spanned by the states which are generated from the standard physical states by the infinitesimal Lorentz boosts. Although the analysis above fails for the massless states $p^2 = 0$ ($p \not\equiv 0$), the result is still the same. One can check it using the explicit formulae for the cohomology states at $p^2 = 0$.

We conclude that the ghost number two cohomology of the extended complex at $p \not\equiv 0$ has the same physical contents as that of the BRST complex.

## 5.2 Lorentz group and the zero momentum states

The Lorentz generators, $J^{\mu\nu}$, act on the zero-momentum states as linear operators of zero $x$-degree:

$$J^{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} \otimes 1 - x^\nu \frac{\partial}{\partial x_\mu} \otimes 1 + 1 \otimes (E^{\mu\nu} + \overline{E}^{\mu\nu}). \tag{5.2.1}$$

| $V^{(0)}_{n+2}$ | $V^{(1)}_{n+1}$ | $V^{(2)}_{n}$ | $V^{(3)}_{n-2}$ | $V^{(4)}_{n-3}$ | $V^{(5)}_{n-4}$ |
|---|---|---|---|---|---|
| $\mathbf{H}_{n+2}$ | $2\,\mathbf{H}_{n+2}$ | $\mathbf{H}_{n+2}$ | | | |
| $\mathbf{H}_{n}$ | $4\,\mathbf{H}_{n}+2\,\mathbf{V}_{n}$ | $5\,\mathbf{H}_{n}+\mathbf{S}_{n}$ $+\mathbf{A}_{n}+2\,\mathbf{V}_{n}$ | $\mathbf{H}_{n}$ | | |
| $\mathbf{H}_{n-2}$ | $4\,\mathbf{H}_{n-2}+2\,\mathbf{V}_{n-2}$ | $6\,\mathbf{H}_{n-2}+\mathbf{S}_{n-2}$ $+\mathbf{A}_{n-2}+4\,\mathbf{V}_{n-2}$ | $5\,\mathbf{H}_{n-2}+\mathbf{S}_{n-2}$ $+\mathbf{A}_{n-2}+2\,\mathbf{V}_{n-2}$ | $2\,\mathbf{H}_{n-2}$ | |
| $\mathbf{H}_{n-4}$ | $4\,\mathbf{H}_{n-4}+2\,\mathbf{V}_{n-4}$ | $6\,\mathbf{H}_{n-4}+\mathbf{S}_{n-4}$ $+\mathbf{A}_{n-4}+4\,\mathbf{V}_{n-4}$ | $6\,\mathbf{H}_{n-4}+\mathbf{S}_{n-4}$ $+\mathbf{A}_{n-4}+4\,\mathbf{V}_{n-4}$ | $2\,\mathbf{H}_{n-4}+2\,\mathbf{V}_{n-4}$ | $\mathbf{H}_{n-4}$ |
| $\mathbf{H}_{n-6}$ | $4\,\mathbf{H}_{n-6}+2\,\mathbf{V}_{n-6}$ | $6\,\mathbf{H}_{n-6}+\mathbf{S}_{n-6}$ $+\mathbf{A}_{n-6}+4\,\mathbf{V}_{n-6}$ | $6\,\mathbf{H}_{n-6}+\mathbf{S}_{n-6}$ $+\mathbf{A}_{n-6}+4\,\mathbf{V}_{n-6}$ | $2\,\mathbf{H}_{n-6}+2\,\mathbf{V}_{n-6}$ | $\mathbf{H}_{n-6}$ |
| ... | ... | ... | ... | ... | ... |

Table 4: Decomposition of the complex (5.2.2) into irreducible representations of $SO(D-1,1)$ for $n \geq 2$.

Consider the complexes

$$0 \longrightarrow V^{(0)}_{n+2} \xrightarrow{d_1} V^{(1)}_{n+1} \xrightarrow{d_1} V^{(2)}_{n} \xrightarrow{d_2} V^{(3)}_{n-2} \xrightarrow{d_1} V^{(4)}_{n-3} \xrightarrow{d_1} V^{(5)}_{n-4} \longrightarrow 0. \tag{5.2.2}$$

which calculate $\mathrm{Gr}H_S(\widehat{V}_0,\widehat{Q})$. By definition $V^{(0)}_n$ and $V^{(5)}_n$ are the spaces of homogeneous polynomials of degree $n$. These spaces are reducible under the Lorentz group because the subspaces of the polynomials of the form $(x^\mu x_\mu)^k h_{n-2k}$ are invariant under $SO(D-1,1)$. Furthermore, if $h_{n-2k}$ are harmonic, $\Box h_{n-2k} = 0$, these subspaces form irreducible representations of $SO(D-1,1)$. We will denote these irreducible representations by $\mathbf{H}_n$. These representations can be alternatively described by Young tableaux as

$$\mathbf{H}_n = \overbrace{\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\cdots}\boxed{\phantom{x}}}^{n}\ . \tag{5.2.3}$$

Now we can write the decomposition of $V^{(0)}_n$ or $V^{(5)}_n$ into irreducible repre-

sentations as

$$V_n^{(0)} = V_n^{(5)} = \mathbf{H}_n + \mathbf{H}_{n-2} + \mathbf{H}_{n-4} + \cdots \ . \tag{5.2.4}$$

At the ghost numbers one and four spaces we find another kind of irreducible representations:

$$\mathbf{V}_n = \begin{array}{l} \overbrace{\square\square\cdots\square\square}^{n}\square \\ \square \end{array}\ , \tag{5.2.5}$$

and, finally in the decompositions of $V_n^{(2)}$ and $V_n^{(3)}$ we will encounter

$$\mathbf{A}_n = \begin{array}{l} \overbrace{\square\square\cdots\square}^{n} \\ \square \\ \square \end{array} \quad \text{and} \quad \mathbf{S}_n = \begin{array}{l} \overbrace{\square\square\cdots\square}^{n} \\ \square\square \end{array}\ . \tag{5.2.6}$$

Suppose $n \geq 2$. Table 4 shows the decomposition of the whole complex (5.2.2) into irreducible representations. From the series of lemmas presented in Appendix B, we know that the complex (5.2.2) has cohomology only in $V_n^{(2)}$. Using Table 4 we conclude that

$$\mathrm{Gr}_{-n} H_S^2(\widehat{V}_0, \widehat{Q}) = \mathbf{H}_n + \mathbf{A}_n + \mathbf{S}_n, \tag{5.2.7}$$

which we obtain by "subtracting" the odd columns from column two and "adding" the even columns. A more detailed analysis shows that if we choose the representatives of $\mathrm{Gr}_{-n} H_S^2(\widehat{V}_0, \widehat{Q})$ so that they belong to $\mathbf{H}_n$, $\mathbf{A}_n$, or $\mathbf{S}_n$, they also will represent cohomology classes of $\widehat{Q}$, no lower $x$-degree corrections required.

Two exceptional cases, $n = 0$ and $n = 1$, have to be treated separately. The decompositions of the complex (5.2.2) into irreducible representations for these the first case is presented in Table 5. Using the results of Appendix B we can infer from Table 5 (left) that

$$\mathrm{Gr}_{-1} H_S^1(\widehat{Q}, \widehat{V}_0) = \mathbf{V}_0 \quad \text{and} \quad \mathrm{Gr}_0 H_S^2(\widehat{Q}, \widehat{V}_0) = \mathbf{H}_0, \tag{5.2.8}$$

and from Table 5 (right) that

$$\mathrm{Gr}_{-1} H_S^2(\widehat{Q}, \widehat{V}_0) = \mathbf{H}_1 + \mathbf{A}_1; \tag{5.2.9}$$

and again, if we pick the representatives of $\mathrm{Gr} H_S$ from the irreducible representations, they will be annihilated by $\widehat{Q}$ and therefore represent the cohomology classes without lower $x$-degree corrections. For this case this can be easily checked by explicit calculation (see Appendix B).

| $V_2^{(0)}$ | $V_1^{(1)}$ | $V_0^{(2)}$ |
|---|---|---|
| $\mathbf{H}_2$ | $2\,\mathbf{H}_2$ | $\mathbf{H}_2$ |
| $\mathbf{H}_0$ | $2\,\mathbf{H}_0 + 2\,\mathbf{V}_0$ | $\mathbf{H}_0+\mathbf{V}_0$ |

| $V_3^{(0)}$ | $V_2^{(1)}$ | $V_1^{(2)}$ |
|---|---|---|
| $\mathbf{H}_3$ | $2\,\mathbf{H}_3$ | $\mathbf{H}_3$ |
| $\mathbf{H}_1$ | $4\,\mathbf{H}_1 + 2\,\mathbf{V}_1$ | $4\,\mathbf{H}_1 + \mathbf{A}_1 + 2\,\mathbf{V}_1$ |

Table 5: Decomposition of the complex (5.2.2) into irreducible representations for $n = 0$ (left) and $n = 1$ (right).

It is tempting to interpret the irreducible representations $\mathbf{H}_n$, $\mathbf{S}_n$, and $\mathbf{A}_n$ as the dilaton, graviton, and antisymmetric tensor. If we do so, it is not quite clear why we have infinitely many irreducible representations for each field, and not just one. We speculate that these representations are related by infinitesimal shifts (the translational part of the Poincaré algebra), which acts on the spaces of polynomials by differentiation with respect to $x^\mu$.

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# Appendix A  Spectral sequence

In this section we review some basic facts about a particular type of the spectral sequence which we use in our analysis of the extended complex. This is not intended to be a complete introduction to the method. Our only goal is to introduce the spaces $E_0$, $E_1$ and $E_2$ equipped with differentials $d_0$, $d_1$ and $d_2$ acting on them. We will prove that these differentials have zero square and provide some motivations to why their cohomologies are related to $\mathrm{Gr}H$. For a more detailed analysis of the first three terms of a spectral sequence, the reader is referred to the book by Dubrovin, Fomenko and Novikov [13]. A general introduction to the spectral sequences from the physicist's point of view and further references can be found in refs. [14, 15].

Let $(C,d)$ be a complex with additional grading $C=\bigoplus C_r$ such that the differential $d$ can be written as

$$d=\partial_0+\partial_1+\partial_2, \tag{A.1}$$

where $\partial_n$ maps $C_r$ to $C_{r+n}$. Since $d$ mixes vectors from different gradings we can not define grading on cohomology $H(d,C)$, but we can still define a decreasing filtration. By filtration of the element $x\in H(d,\mathrm{Gr}C)$ we will mean the smallest (negative) integer $s$ such that $x$ is representable by a cocycle

$$\overline{x}=x_r+x_{r+1}+\cdots, \tag{A.2}$$

where $x_r\in C_r$. We will denote the space of such vectors by $F_rH(d,C)$. Using the filtration we can define a graded space associated to the cohomology $H(d,C)$

$$\mathrm{Gr}H=\bigoplus_s \mathrm{Gr}_sH, \quad \text{where} \quad \mathrm{Gr}_sH=F_sH/F_{s+1}H. \tag{A.3}$$

The investigation of the spaces $\mathrm{Gr}_sH^n$ is carried out using the method of "successive approximations" based on what is called the "spectral sequence". The idea is to construct a sequence of complexes $(E_n,d_n)$ such that $E_{n+1}=H(d_n,E_n)$ which converges to $\mathrm{Gr}H$

$$\mathrm{Gr}_sH^n=E_\infty^{s,n-s}. \tag{A.4}$$

Differentials $d_n$ are acting on the spaces $E_n^{r,s}$ as follows

$$d_n: \quad E_n^{r,s}\to E_n^{r+n,s-n+1} \tag{A.5}$$

For a complete description of the spectral sequence and the proof of the theorem which states that the spectral sequence converges to Gr$H$ we refer to [13, 17]. Let us describe the first few terms of the spectral sequence. Suppose $\overline{x}$ given in Eq. (A.2) represent a cohomology class $x$ in $F_sH(d,C)$. Applying $d = \partial_0 + \partial_1 + \partial_2$ to $\overline{x}$ we obtain

$$\begin{aligned} d\overline{x} = \partial_0\overline{x}_r + (\partial_1\overline{x}_r + \partial_0\overline{x}_{r+1}) + (\partial_2\overline{x}_r + \partial_1\overline{x}_{r+1} + \partial_0\overline{x}_{r+2}) \\ +(\partial_2\overline{x}_{r+1} + \partial_1\overline{x}_{r+2} + \partial_0\overline{x}_{r+3}) + \cdots . \end{aligned} \tag{A.6}$$

where we enclosed in braces the terms from the same $C_r$ space. It follows that

$$\partial_0\overline{x}_r = 0, \quad \partial_1\overline{x}_r = -\partial_0\overline{x}_{r+1}, \quad \partial_2\overline{x}_r = -\partial_1\overline{x}_{r+1} - \partial_0\overline{x}_{r+2}, \ldots, \tag{A.7}$$

form which we conclude that $\overline{x}_r$ is a $\partial_0$ cocycle and a $\partial_1$ cocycle modulo image of $\partial_0$. This suggests that the first approximation in the spectral sequence should be $E_1 = H(\partial_0, C)$ and $d_0 = \partial_0$, the second approximation is $E_2 = H(d_1, E_1)$, where $d_1$ is induced on $E_1 = H(\partial_0, C)$ by $\partial_1$. Using the second equation of (A.7) we can formally find $\overline{x}_{r+1}$ in terms of $\overline{x}_r$ as $\overline{x}_{r+1} = -\partial_0^{-1}\partial_1\overline{x}_r$ and rewrite the last equation of (A.7) as

$$(\partial_2 - \partial_1\partial_0^{-1}\partial_1)\overline{x}_r = -\partial_0\overline{x}_{r+2}. \tag{A.8}$$

The last formulae suggests that $d_2$ is induced on $E_2 = H(d_1, E_1)$ by $\partial_2 - \partial_1\partial_0^{-1}\partial_1$ and the third approximation is $E_3 = H(d_2, E_2)$.

Let us show that these differentials are well defined and square to zero. For $d_0$ the first is obvious since it acts on the same space as $\partial_0$ and $\partial_0^2 = 0$ follows from $d^2 = 0$ which is equivalent to

$$\partial_0^2 = \{\partial_0, \partial_1\} = \partial_1^2 + \{\partial_0, \partial_2\} = \partial_2^2 = 0. \tag{A.9}$$

In order to show that $d_1$ is well defined we have to show that $\partial_1$ maps $\partial_0$-closed vectors to $\partial_0$-closed vectors and $\partial_0$-trivial to $\partial_0$-trivial. This easily follows from the anticommutation relation $\{\partial_0, \partial_1\} = 0$. Let us prove that $d_1^2 = 0$. Suppose $x \in E_1$ can be represented by a cocycle $\overline{x} \in C$, $\partial_0\overline{x} = 0$. Then applying $\partial_1^2 = -\{\partial_0, \partial_2\}$ to $\overline{x}$ we obtain a trivial cocycle $\partial_1^2\overline{x} = -\partial_0\partial_2\overline{x}$. In cohomology this implies that $d_1^2 x = 0$.

Before we consider the differential $d_2$, let us describe the space $E_2$ on which it acts in greater details. By definition $E_2 = H(d_1, E_1)$, but $E_1$ in turn is the cohomology of the original complex with respect to $\partial_0$. Therefore, in order to find a $d_1$ cocycle we should start with a $\partial_0$ cocycle $\overline{x}$ and require that its image under $\partial_1$ is $\partial_0$ exact. Two cocycles $\overline{x}$ and $\overline{x}'$ represent the same $d_1$ cohomology class if $\overline{x}' - \overline{x} \in \operatorname{Im}\partial_0 + \partial_1 \ker \partial_0$. Let $\overline{x}$ be a $d_1$ cocycle, which means that

$$\partial_0 \overline{x} = 0 \qquad \text{and} \qquad \partial_1 \overline{x} = \partial_0 \overline{y}. \tag{A.10}$$

We define $d_2 \overline{x}$ as

$$d_2 \overline{x} = \partial_2 \overline{x} - \partial_1 \overline{y} = (\partial_2 - \partial_1 \partial_0^{-1} \partial_1) \overline{x}. \tag{A.11}$$

Let us show that the result is again a $d_1$ cocycle. Indeed, using the properties of $\partial_n$ listed in Eq. (A.9) we obtain

$$\partial_0 d_2 \overline{x} = \partial_0 \partial_2 \overline{x} + \partial_0 \partial_1 \overline{y} = \partial_0 \partial_2 \overline{x} - \partial_1 \partial_0 \overline{y} = \{\partial_0, \partial_2\} \overline{x} - \partial_1^2 \overline{x} = 0$$

and

$$\partial_1 d_2 \overline{x} = \partial_1 \partial_2 \overline{x} - \partial_1^2 \overline{y} = \partial_0 \partial_2 \overline{y}.$$

Similarly we can prove that if $\overline{x}$ and $\overline{x}'$ belong to the same $d_1$ cohomology class then their $d_2$ images belong to the same class as well. This will finally establish the correctness of the definition of $d_2$ as an operator on $E_2$. In conclusion let us show that $d_2^2 = 0$. With $\overline{x}$ as above we have

$$\begin{aligned} d_2^2 \overline{x} =& \partial_2(\partial_2 \overline{x} - \partial_1 \overline{y}) - \partial_1 \partial_0^{-1} \partial_1 (\partial_2 \overline{x} - \partial_1 \overline{y}) \\ =& \partial_1 \partial_2 \overline{y} + \partial_1 \partial_0^{-1} \partial_2 \partial_1 \overline{x} - \partial_1 \partial_0^{-1} \partial_0 \partial_2 \overline{y} - \partial_1 \partial_0^{-1} \partial_2 \partial_0 \overline{y} \\ =& \partial_1 \partial_2 \overline{y} - \partial_1 \partial_2 \overline{y} = 0 \end{aligned}$$

which completes our analysis of $d_2$.

# Appendix B Three lemmas

In this appendix we will calculate the cohomology of the following complex

$$0 \to V^{(0)} \xrightarrow{d^{(0)}} V^{(1)} \xrightarrow{d^{(1)}} V^{(2)} \xrightarrow{d^{(2)}} V^{(3)} \xrightarrow{d^{(3)}} V^{(4)} \xrightarrow{d^{(4)}} V^{(5)} \to 0, \tag{B.1}$$

where $V^{(0)} \simeq V^{(0)} \simeq \mathbb{C}[x^0 \cdots x^{D-1}]$, $V^{(0)} \simeq V^{(0)} \simeq \mathbb{C}^{2D}[x^0 \cdots x^{D-1}]$ and $V^{(0)} \simeq V^{(0)} \simeq \mathbb{C}^{D^2+1}[x^0 \cdots x^{D-1}]$. Following the notations of section 4.3 we represent the elements of $V^{(0)}$ and $V^{(5)}$ as $(R^{[0]})$ and $(R^{[5]})$, elements of $V^{(2)}$ and $V^{(4)}$ as $(Q^{[2]}_{\mu\nu}, R^{[2]})$ and $(Q^{[3]}_{\mu\nu}, R^{[3]})$, and elements of $V^{(2)}$ and $V^{(4)}$ by $(P_\mu, \overline{P}_\mu)$. Differentials $d^{(n)}$ act as follows

$$
\begin{array}{rcl}
(R^{[0]}) & \overset{d^{(0)}}{\searrow} & (0) \\
(P^{[1]}_\mu,\ \overline{P}^{[1]}_\mu) & \overset{d^{(1)}}{\searrow} & (\partial_\mu R^{[0]},\ \partial_\mu R^{[0]}) \\
(Q^{[2]}_{\mu\nu},\ R^{[2]}) & \overset{d^{(2)}}{\searrow} & (\partial_\mu \overline{P}^{[1]}_\nu - \partial_\nu P^{[1]}_\mu,\ \partial^\mu \overline{P}^{[1]}_\mu - \partial^\mu P^{[1]}_\mu) \\
(Q^{[3]}_{\mu\nu},\ R^{[3]}) & \overset{d^{(3)}}{\searrow} & (\delta Q^{[3]}_{\mu\nu},\ \delta R^{[3]}) \\
(P^{[4]}_\mu,\ \overline{P}^{[4]}_\mu) & \overset{d^{(4)}}{\searrow} & (\partial^\nu Q^{[3]}_{\mu\nu} + \partial_\mu R^{[3]},\ \partial^\nu Q^{[3]}_{\nu\mu} + \partial_\mu R^{[3]}) \\
(R^{[5]}) & & (\partial_\mu P^{[4]} - \partial_\mu \overline{P}^{[4]})
\end{array} \tag{B.2}
$$

where

$$
\delta Q^{[3]}_{\mu\nu} = \Box Q^{[2]}_{\mu\nu} - \partial^\lambda \partial_\mu Q^{[2]}_{\lambda\nu} - \partial^\lambda \partial_\nu Q^{[2]}_{\mu\lambda} + 2\partial_\mu \partial_\nu R^{[2]} \tag{B.3}
$$

$$
\delta R^{[3]} = -\partial^\lambda \partial^\rho Q^{[2]}_{\lambda\rho} + 2\Box R^{[2]} \tag{B.4}
$$

It is obvious that $d^{(4)}$ is surjective and the kernel of $d^{(0)}$ contains only constant polynomials. Thus we conclude that $H^0 = \mathbb{C}$ and $H^5 = 0$. The other cohomology spaces are described by the following lemmas

**Lemma 1** *$H^1$ is finite dimensional and* $\dim H^1 = \frac{D(D+1)}{2}$. *$H^1$ cohomology classes can be represented by polynomials of degree no bigger than one.*

**Proof.** According to (B.2) $H^1$ is a quotient of the space $S$ of solutions to the system of first order differential equations

$$
\begin{cases} \partial_\mu P_\nu = \partial_\nu \overline{P}_\mu \\ \partial^\mu P_\mu = \partial^\mu \overline{P}_\mu \end{cases} \tag{B.5}
$$

by the space $T$ of trivial solutions $P_\mu = \overline{P}_\mu = \partial_\mu R$. Note that both $S$ and $T$ naturally decompose into direct sum of the spaces of homogeneous polynomials and so does the quotient

$$
S = \bigoplus S^n, \quad T = \bigoplus T^n, \quad H^1 = S/T = \bigoplus S^n/T^n \tag{B.6}
$$

We want to prove that $S^n = T^n$ for $n > 1$. Let $P_\mu$ and $\overline{P}_\mu$ be homogeneous polynomials of degree $n > 1$ that satisfy Eq. (B.5). First, it is obvious that $P_\mu = 0$ for every $\mu$ requires $\overline{P}_\mu = 0$. Indeed, if $P_\mu = 0$ for every $\mu$ then according to the first equation in Eq. (B.5) $\partial_\nu \overline{P}_\mu = 0$ for every $\mu$ and $\nu$ and since by assumption $\deg \overline{P}_\mu > 1$ this means $\overline{P}_\mu = 0$. Second, using the first equation of (B.5) twice we obtain

$$\partial_a \partial_\nu P_\mu = \partial_\alpha \partial_\mu \overline{P}_\nu = \partial_\mu \partial_\alpha \overline{P}_\nu = \partial_\alpha \partial_\mu P_\mu. \tag{B.7}$$

Therefore for any $\alpha$, $\mu$ and $\nu$

$$\partial_\mu(\partial_\alpha P_\mu - \partial_\mu P_\alpha) = 0. \tag{B.8}$$

And since $\deg(\partial_\alpha P_\mu - \partial_\mu P_\alpha) = n - 1 > 0$ we conclude that $\partial_\alpha P_\mu - \partial_\mu P_\alpha = 0$ and there exist $R$ such that $P_\mu = \partial_\mu R$. Subtracting a trivial solution $(\partial_\mu R, \partial_\mu R)$ from $(P_\mu, \overline{P}_\mu)$ we get another solution $(P'_\mu = 0, \overline{P}'_\mu = \overline{P}_\mu - \partial_\mu R)$. According to our first observation $P'_\mu = 0$ requires $\overline{P}'_\mu = 0$ and thus $\overline{P}_\mu = P_\mu = \partial_\mu R$.

It is easy to see that there are exactly $\frac{D(D-1)}{2}$ non-trivial solutions of degree one and $D$ non-trivial constant solutions which can be written as

$$P_\mu = -\overline{P}_\mu = \xi_{[\mu\nu]} x^\nu, \quad \text{and} \quad P_\mu = -\overline{P}_\mu = const \tag{B.9}$$

where $\xi_{[\mu\nu]}$ is an antisymmetric tensor.

**Lemma 2** $H^3 = 0$

**Proof.** This is the most difficult lemma in this work. We have to show that any solution of the system

$$\begin{aligned} \partial^\nu Q^{[3]}_{\mu\nu} + \partial_\mu R^{[3]} &= 0 \\ \partial^\nu Q^{[3]}_{\nu\mu} + \partial_\mu R^{[3]} &= 0 \end{aligned} \tag{B.10}$$

can be represented in the form

$$\begin{aligned} \delta Q^{[3]}_{\mu\nu} &= \Box Q^{[2]}_{\mu\nu} - \partial^\lambda \partial_\mu Q^{[2]}_{\lambda\nu} - \partial^\lambda \partial_\nu Q^{[2]}_{\mu\lambda} + 2\partial_\mu \partial_\nu R^{[2]} \\ \delta R^{[3]} &= -\partial^\lambda \partial^\rho Q^{[2]}_{\lambda\rho} + 2\Box R^{[2]} \end{aligned} \tag{B.11}$$

We will start from an arbitrary solution of the system (B.10) and will be modifying it step by step by adding the trivial solutions of the form (B.11) in order to get zero.

We can use the same arguments as in lemma 1 to consider only homogeneous polynomials of some degree $m$. Suppose $m \geq 1$. We will describe an iterative procedure which will allow us to modify $(Q^{[3]}_{\mu\nu},\ R^{[3]})$ so that $Q^{[3]}_{\mu\nu}$ will depend only on one variable, say $x^0$ and its only nonzero components be $Q^{[3]}_{\mu 0}$ and $Q^{[3]}_{0\nu}$.

If this is the case, the cocycle condition (Eq. (B.10)) tells us that

$$\partial_\mu R = \partial_0 Q_{\mu,0} = \partial_0 Q_{0,\mu}, \tag{B.12}$$

moreover, $Q_{\mu,0} = C_\mu x_0^m$ and $Q_{0,\mu} = \overline{C}_\mu x_0^m$. Furthermore, using Eq. (B.12) we conclude that $C_\mu = \overline{C}_\mu$. Integrating Eq. (B.12) we obtain

$$R^{[3]} = \sum_{i=1}^{D-1} mC_i x_i x_0^{m-1} + C_0 x_0^{26}. \tag{B.13}$$

One can check that such solution $(Q^{[3]}_{\mu\nu},\ R^{[3]})$ can be written in the form (B.11) with $Q^{[2]}_{\mu,\nu} = 0$ and

$$R^{[2]} = \frac{1}{2} \sum_{i=1}^{D-1} \left( \frac{C_1}{m+1} x_i x_0^{m+1} + \frac{C_0}{(m+1)(m+2)} x_0^{m+2} \right).$$

Now let us describe the procedure which reduces any solution to the abovementioned form. Our first objective is to get rid of $x_i$ dependence for $i = 1..D-1$. Let us pick $i$. The following four step algorithm will make $(Q_{\mu\nu}, R)$ independent of $x_i$. We will see that when we apply the procedure to $(Q_{\mu\nu}, R)$ which does not depend on some other $x_k$ it will not introduce $x_k$ dependence in the output. This observation will allow us to apply the algorithm $D-1$ times and make $(Q_{\mu\nu}, R)$ depend only on $x_0$.

**Step 1** Let us introduce some notations. For a polynomial $P$ we will denote the minimal degree of $x_i$ among all the monomials in $P$ by $n_i(P)$. For a zero polynomial we formally set $n_i(0) = +\infty$. Given a matrix of polynomials $Q_{\mu,\nu}$, let

$$N_i(Q_{\mu\nu}) = \min_{\substack{\mu \neq i \\ \nu \neq i}} n_i(Q_{\mu\nu}) \tag{B.14}$$

Since $Q^{[3]}_{\mu\nu}$ are homogeneous polynomials of degree $m$ we can write them as

$$Q^{[3]}_{\mu\nu} = \sum_{m_0+\cdots+m_{D-1}=m} C_{m_0\cdots m_{D-1},\,\mu\nu} x_0^{m_0}\cdots x_{D-1}^{m_{D-1}} \tag{B.15}$$

Let us show that it is possible to add a trivial solution to $(Q_{\mu\nu}, R)$ and increase $N_i(Q_{\mu\nu})$ by one. Indeed, let

$$Q^{[2]}_{\mu\nu} = \begin{cases} \displaystyle\sum_{m_0+\cdots+m_{D-1}=m} -\frac{C_{m_0\cdots m_{D-1},\,\mu\nu}\, x_0^{m_0}\cdots x_i^{m_i+2}\cdots x_{D-1}^{m_{D-1}}}{(m_i+1)(m_i+2)} & \text{for } \mu,\nu\neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and $R^{[2]} = 0$.

It is easy to see that

$$N_i(Q^{[3]}_{\mu\nu} + \delta Q^{[3]}_{\mu\nu}) \geq N_i(Q^{[3]}_{\mu\nu}) + 1$$

where $\delta Q^{[3]}_{\mu\nu}$ comes from the trivial solution generated by $(Q^{[2]}_{\mu\nu}, R^{[2]})$ according to (B.11). Repeating this procedure, we will increase $N_i(Q^{[3]}_{\mu\nu})$ at least by one every time. Since $Q^{[3]}_{\mu\nu}$ are homogeneous polynomials of degree $m$, $N_i$ is either less then $m+1$ or equal $+\infty$. Therefore after a finite number of steps we will make $N_i = +\infty$ which means that all $Q^{[3]}_{\mu\nu}$ are zero for $\mu\neq i$ and $\nu\neq i$.

**Step 2** Since $(Q^{[3]}_{\mu\nu}, R^{[3]})$ is a solution to the system (B.10) we can write

$$\begin{aligned} \partial_i Q^{[3]}_{\mu i} &= \partial_\mu R^{[3]} = \partial_i Q^{[3]}_{i\mu}, \quad \mu\neq i \\ \partial^\nu Q^{[3]}_{\nu i} &= \partial_i R^{[3]} = \partial^\nu Q^{[3]}_{i\nu}. \end{aligned}$$

Suppose

$$R^{[3]} = \sum_{m_0+\cdots+m_{D-1}=m} D_{m_0\cdots m_{D-1}} x_0^{m_0}\cdots x_{D-1}^{m_{D-1}},$$

then we choose $R^{[2]} = 0$,

$$Q^{[2]}_{ii} = \sum_{m_0+\cdots+m_{D-1}=m} \frac{D_{m_0\cdots m_{D-1}}}{(m_i+1)(m_i+2)} x_0^{m_0}\cdots x_i^{m_i+2}\cdots x_{D-1}^{m_{D-1}},$$

and $Q^{[2]}_{\mu\nu} = 0$ for all the other $\mu$ and $\nu$. It is easy to see that $\widetilde{Q}^{[3]}_{\mu\nu} = Q^{[3]}_{\mu\nu} + \delta Q^{[3]}_{\mu\nu}$ have the following properties:

– all $\widetilde{Q}^{[3]}_{\mu\nu}$ except $\widetilde{Q}_{ii}$ do not depend on $x_i$

– $\widetilde{Q}^{[3]}_{\mu\nu} = 0$ for all $\mu \neq i$ and $\nu \neq i$.

**Step 3** Suppose $(Q^{[3]}_{\mu\nu}, R^{[3]})$ is of the form we obtained at the end of step 2. Since $\partial_i Q^{[3]}_{\mu i} = 0$ for $\mu \neq i$, then $\partial_\mu R^{[3]} = 0$ for $\mu \neq i$ and therefore, $R^{[3]}$ depends only on $x_i$. Thus there exists $R^{[2]}$ such that $R^{[3]} = -2\partial_i^2 R^{[2]}$ and $R^{[2]}$ depends only on $x_i$. adding a trivial solution generated by $(0, R^{[2]})$ we can make $R^{[3]} = 0$.

**Step 4** Using $R^{[3]} = 0$ we can rewrite the system (B.10) as follows.

$$\partial_i Q^{[3]}_{ii} = -\sum_{\nu \neq i} \partial^\nu Q^{[3]}_{\nu i}.$$

Therefore, $Q^{[3]}_{ii} = Q^{[3]}_{ii,0} + x_i Q^{[3]}_{ii,1}$, where $Q^{[3]}_{ii,0}$ and $Q^{[3]}_{ii,1}$ do not depend on $x_i$. For every $Q^{[3]}_{ii,1}$ we can find a polynomial $P$ which depends on the same set of variables and $\Box P = -Q^{[3]}_{ii,1}$. Choose $Q^{[2]}_{ii} = x_i\, P$, $R^{[2]} = 0$ and $Q^{[2]}_{\mu\nu} = 0$ for $(\mu,\nu) \neq (i,i)$. Adding the corresponding trivial solution we achieve that $Q^{[3]}_{\mu\nu}$ and $R^{[3]}$ do not depend on $x_i$.

Repeating this program $D-1$ times for each value of $i = 1, \ldots, D-1$, we make $(Q^{[3]}_{\mu\nu}, R^{[3]})$ depend only on $x_0$. Now we can repeat the first step once again with $i = 0$ and make $Q^{[3]}_{kj} = 0$ for $k, j = 1, \ldots D-1$. We have already proven that such solution is trivial.

Recall that in the very beginning of our analysis we have made an assumption that the polynomials have non-zero degree ($m \geq 1$). Therefore we have to consider this last case separately. If polynomials $Q^{[3]}_{\mu\nu}$ and $R^{[3]}$ are constant, they trivially satisfy the system (B.10). To show that any such constant solution can be represented in the form (B.11), it is sufficient to take $Q^{[2]}_{\mu\nu} = q_{\mu\nu,0}(x_0)^2/2 + q_{\mu\nu,1}(x_1)^2/2$, which generates $(Q^{[3]}_{\mu\nu}, R^{[3]})$ if $q_{\mu\nu,0}$ and $q_{\mu\nu,1}$ are chosen so that

$$\begin{aligned}
q_{00,1} - q_{00,0} &= Q^{[3]}_{00}, \\
q_{11,0} - q_{11,0} &= Q^{[3]}_{11}, \\
q_{00,1} + q_{00,0} + q_{11,1} + q_{11,0} &= R^{[3]}, \\
q_{0\nu,1} &= Q^{[3]}_{0\nu} \text{ for } \nu \neq 0,
\end{aligned}$$

$$
\begin{aligned}
q_{\mu 0,1} &= Q^{[3]}_{\mu 0} \text{ for } \mu \neq 0,\\
q_{1\nu,0} &= Q^{[3]}_{1\nu} \text{ for } \nu \neq 1,\\
q_{\mu 1,0} &= Q^{[3]}_{\mu 1} \text{ for } \mu \neq 1,\\
q_{\mu\nu,0} + q_{\mu\nu,1} &= Q^{[3]}_{\mu\nu} \text{ for } \mu, \nu > 1.
\end{aligned}
$$

This completes the proof of Lemma 2.

**Lemma 3** $H^4 = 0$.

**Proof.** It is almost obvious that the image of $d^{(3)}$ covers the whole kernel of $d^{(4)}$ in $V^{(4)}$ because the space $V^{(3)}$ is much bigger than $V^{(4)}$ at every degree. Indeed we will show that it is sufficient to consider a subspace of $V^{(3)}$ spanned by the zeroth row and the zeroth column of the matrix $Q_{\mu\nu}$. Loosely speaking the row will cover $P_\mu$ and the column will cover $\overline{P}_\mu$.

Suppose $(P_\mu, \overline{P}_\mu) \in \ker d^{(4)}$ or equivalently

$$
d^{(4)}(P_\mu, \overline{P}_\mu) = (\partial^\mu P_\mu - \partial^\mu \overline{P}_\mu) = 0 \tag{B.16}
$$

we want to show that subtracting vectors of the form $d^{(3)}(Q_{\mu\nu}, 0)$ from $(P_\mu, \overline{P}_\mu)$ we can get reduce it to zero. First of all we can easily get rid of the spacial components $P_i$ and $\overline{P}_i$ for $i = 1 \cdots D-1$ using $Q_{\mu\nu}$ which has the only non-zero components given by

$$
Q_{0i} = \int P_i dx^0 \quad \text{and} \quad Q_{i0} = \int \overline{P}_i dx^0 + C_i(x^1 \cdots x^{D-1}) \tag{B.17}
$$

This will reduce $P_\mu$ and $\overline{P}_\mu$ to the form $P_\mu = a\delta_{0,\mu}$ and $\overline{P}_\mu = \overline{a}\delta_{0,\mu}$. Furthermore, according to Eq. (B.16) polynomials $a$ and $\overline{a}$ have the same derivative with respect to $x^0$. Thus varying say $C_1(x^1 \cdots x^{D-1})$ in Eq. (B.17) we can achieve that $a = \overline{a}$. Finally if we have a vector given by $P_\mu = \overline{P}_\mu = a\delta_{0,\mu}$ we can use $Q_{00}$ to reduce it to zero.

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